

## FACE-SUBGROUPS OF PERMUTATION POLYTOPES

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## 1. THE CONJECTURE

A permutation polytope  $P(G)$  is the convex hull, in  $\mathbb{R}^{n \times n}$ , of a subgroup  $G$  of the group  $S_n$  of  $n \times n$  permutation matrices. This is a convex geometric invariant of a permutation representation, and it yields various numerical invariants like dimension, volume, diameter,  $f$ -vector, etc. Permutation polytopes and their faces were studied in [1].

Let  $[n] := \{1, \dots, n\} = \bigsqcup I_k$  be a partition of the ground set. We define the stabilizer of this partition as

$$\text{stab}(G; (I_k)_k) := \{\sigma \in G : \sigma(I_k) = I_k \text{ for all } k\} \leq G.$$

**Lemma 1.**  $P(\text{stab}(G; (I_k)_k))$  is a face of  $P(G)$ .

In [1] we formulated the following conjecture.

**Conjecture 2.** Let  $G \leq S_n$ . Suppose  $H \leq G$  is a subgroup such that  $P(H) \preceq P(G)$  is a face. Then there is a partition  $[n] = \bigsqcup I_k$  so that  $H = \text{stab}(G; (I_k)_k)$ .

As evidence for the conjecture, we observed that it holds for  $G = S_n$  as well as for cyclic  $G$ . Later, Jessica Nowack and Daniel Heinrich verified the conjecture for the dihedral groups  $D_n$ , compare [3, Prop. 6.1].

Subsequent efforts to better understand the inequality description of permutation polytopes, even of abelian groups, can lead one to believe that there is hardly anything one could say about general permutation polytopes [2].

## 2. THE PROOF

For  $i \in [n]$  we denote the orbit under  $G$  by  $G.i$ , and the stabilizer of  $i$  by  $G_i$ .

**Lemma 3.** Let  $A := \frac{1}{|G|} \sum_{g \in G} g$  be the vertex-barycenter of  $G \subset \mathbb{R}^{n \times n}$ . Then  $A$  has entries

$$a_{ij} = \begin{cases} \frac{1}{|G.i|} & \text{if } G.i = G.j \\ 0 & \text{else.} \end{cases}$$

In particular,  $A$  only depends on the orbit structure of  $G$ .

*Proof.* The  $ij$ -entry of  $\sum_{g \in G} g$  counts  $g \in G$  with  $g(j) = i$ . This number is either zero or equal to  $|G_i|$ . Now use  $|G| = |G_i| \cdot |G.i|$ .  $\square$

**Theorem 4.** Conjecture 2 is true.

*Proof.* Let  $H \leq G$  be a subgroup such that  $P(H) \preceq P(G)$  is a face. Let  $[n] = \bigsqcup I_k$  be the partition of  $[n]$  into  $H$ -orbits, and set  $\hat{H} := \text{stab}(G; (I_k)_k)$ . Then  $H$  and  $\hat{H}$  have the same orbit partition. Thus,  $P(H)$  and  $P(\hat{H})$  are faces of  $P(G)$  by assumption and by Lemma 1, respectively. They share a relative interior point by Lemma 3. This implies  $P(H) = P(\hat{H})$ .  $\square$

## REFERENCES

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